Vector Fields, Flows and Lie Bracket

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Exercise 1. Let A, B be two matrices in $M_n(\mathbb{R})$. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ and $Y : \mathbb{R}^n \to \mathbb{R}^n$ be the vector fields defined by

$$X(p) = Ap, \quad Y(p) = Bp.$$

Compute $[X, Y]$.

Exercise 2 (Lie Groups)**.** Let G be a Lie group. For all $g \in G$, let $L_g : G \to G$ be the left multiplication by g. A vector field $X \in \mathcal{X}(G)$ is said to be *left-invariant* if for all g in G, $(L_g)_* X = X$. Let $\mathcal{L}(G)$ be the set of left-invariant vector fields on G.

1. If $X, Y \in \mathcal{L}(M)$, prove that $[X, Y] \in \mathcal{L}(G)$.
2. Prove that the map $\varphi : \mathcal{L}(G) \to T_e G$ defined by $\varphi(X) = X(e)$ is a linear isomorphism. As an application, deduce that G is parallelizable.
3. Compute $\mathcal{L}(G)$ when $G = GL_n(\mathbb{R})$, $G = SL_n(\mathbb{R})$, $G = O_n(\mathbb{R})$.

Exercise 3. Let M be a manifold and X a vector field on M such that for all vector fields Y on M, $[X, Y] = 0$. What can you say about X ?

Exercise 4 (Transitivity of Diff(M))**.**

1. Let a and b be two points in the open ball $\mathbb{B} = \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$. Prove that there exists a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a) = b$ and $f \equiv id$ outside $\mathbb{B}$.
2. Let M be a connected manifold. Prove that Diff(M) acts transitively on M.
3. Is this action k transitive (for $k \geqslant 1$)?

Exercise 5 (Hessian)**.** Let M be a manifold, and $f : M \to \mathbb{R}$ a smooth function. Let p be a critical point of f. The Hessian $Hess(f)_p$ of f at p is the map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ defined by $Hess(f)_p(X, Y) = X(Yf)(p)$.

1. Prove that $Hess(f)_p$ is bilinear, symmetric.
2. Prove that $Hess(f)_p(X, Y)$ depends only on the values of X and Y at p: if X' and Y' are two vector fields such that $X'(p) = X(p)$ and $Y'(p) = Y(p)$, then $Hess(f)_p(X, Y) = Hess(f)_p(X', Y')$.

Exercise 6 (Pseudo-gradient)**.** Let $f : M \to \mathbb{R}$ be a smooth function defined on a smooth manifold M. A *pseudo-gradient* of f is a vector field X of M such that, for all $x \in M \setminus \mathrm{Crit}(f)$, $df(x) \cdot X(x) > 0$.

1. Let $M \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Use the gradient of $\widetilde{f}$ to produce a pseudo-gradient of $\widetilde{f}|_M$.
2. Let M be a submanifold of $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ be the projection on the first coordinate ("height function"). What are the critical points of f? Give an integrable pseudo-gradient.
3. Show the existence of pseudo-gradients in the general case.

4. Given an integrable pseudo-gradient X of $f : M \to \mathbb{R}$, let $(\phi_t)_{t\in\mathbb{R}}$ be its flow. For all $x \in M$, show that $t \mapsto f \circ \phi_t(x)$ is increasing and that

$$\bigcap_{T>0} \overline{\{\phi_t(x) \mid t > T\}} \subset \mathrm{Crit}(f).$$

5. Suppose that f has only isolated critical points, show that, for all $x \in M$, there exist critical points $\alpha, \omega \in M$ such that $\phi_t(x) \to \alpha$ as $t \to -\infty$ and $\phi_t(x) \to \omega$ as $t \to +\infty$.

6. Let M be a closed submanifold of $\mathbb{R}^n$, and $f : M \to \mathbb{R}$ a smooth function. Let $M_t = \{x \in M \mid f(x) \leqslant t\}$. Let $a, b \in \mathbb{R}$ be such that $M \cap f^{-1}([a, b])$ and $\mathrm{Crit}(f)$ are disjoint. Prove that M_a and M_b are diffeomorphic.